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The Deflection of Charged Particles By a Current-Carrying Plasma+

by

Glenn Joyce, David Montgomery,
and Celso Roqué*

Department of Physics and Astronomy
University of Iowa
Iowa City, Iowa

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*Rockefeller Foundation Fellow.

ABSTRACT

Previously derived formal expressions for the mean square deflection of a beam of energetic test particles from a spatially uniform plasma are evaluated numerically. The plasma electrons are assumed to be drifting relative to the ions with a velocity which ranges between zero and the critical drift velocity for the onset of the ion sound wave instability. Though the scattering angle diverges at the stability boundary, one must be able to make the drift velocity prohibitively close to the critical drift velocity (to within a small fraction of a percent of it) to see a marked enhancement. In no case was the scattering enhanced over the thermal value by as much as a factor of 10.

I. INTRODUCTION

One analytical framework in which the scattering of energetic test particles by a plasma can be discussed is provided by the auto-correlation function for the microscopic electric field. In a spatially-uniform plasma in which there exist no macroscopic charge densities and in which the mean electric field $\langle \vec{E}(\vec{x}, t)\rangle$ vanishes, the auto-correlation $\langle \vec{E}(\vec{x}, t)\, \vec{E}(\vec{x}', t')\rangle$ is non-zero, and its Fourier transform in space and time, $\langle \overset{\Rightarrow}{S} \rangle_{\vec{k}\omega}$, directly determines the small-angle scattering of a tenuous beam of test charges which is fired through the plasma.

It has been previously pointed out that the angular deflections of ion beams are measurable quantities with attainable laboratory plasmas.[1] This calculation concerns a more detailed description of the scattering to be expected in a specific experimental situation: a uniform plasma carrying a current with electrons which are hot relative to the ions. The relative drift of electrons with respect to ions varies between zero and the critical drift velocity for the onset of the ion sound wave instability.[2,3]

That enhanced scattering is to be expected in this situation was first made clear by Tidman and Eviatar[4] and Eviatar,[5] who discussed the problem from the point of view of the Fokker-Planck coefficients derived from the Balescu-Lenard equation for the test particle beam. Reference 4 deals with the isotropic (zero drift velocity) case, and Reference 5 is restricted to the case of parallel drift velocity and test particle velocity. This work generalizes to the case of an arbitrary angle of incidence, and finds the most enhanced scattering at large angles between the drift and test particle velocities. Also new in the present calculation is an attempt to avoid any *ad hoc* estimates of the multi-dimensional integrals which appear; use is made of numerical evaluation of the integrals when the need arises.

One result emerges from this calculation which is somewhat disappointing. It has often been observed (see, for example, Rostoker[6]) that fluctuations in the microscopic electric field are greatly enhanced at the threshold of linear instability, and diverge at the boundary. The effect is well known in the theory of *incoherent electromagnetic wave scattering*.[7,8] In the present case, one expects the effect to be reflected by a divergence, at the linear stability boundary, of the mean square scattering angle. We do in fact find such a divergence (Section III), but it is only logarithmic in its dependence on the difference between the drift velocity and the critical drift velocity for the onset of the instability. In practice, this turns out to mean that if one wants the scattering to be enhanced by as much as an order of magnitude over the zero-drift value, the critical drift velocity must be approached within a very small fraction of a per cent. Experimentally, of course, it is at present out of the question to control the drift velocity to this accuracy. The enhancement of scattering is much less dramatic than in the electromagnetic wave case.

An outline of the work is as follows. The proposed experimental geometry is described, and the integral expressions for the mean square scattering angles are derived in Section II. Evaluation of these integrals is described in Section III, and a short summary is provided in Section IV. An Appendix relates the expression for small-angle scattering which we have used to the estimated ninety-degree deflection time of Tidman and Eviatar, and speculates on the range of validity of the latter.

II. EXPRESSIONS FOR THE SCATTERING COEFFICIENTS

The experimental arrangement is idealized as in Figure 1. The plasma is treated as a uniform rectangular slab of infinite length and thickness d. The electron and ion distributions are treated as uniform and field-free, in the macroscopic sense. When there is a relative drift of the electrons, $\vec{V}_d$, it is parallel to the axis of the slab. The test particles have velocity $\vec{V}_o$, and strike the plasma with an angle γ between $\vec{V}_o$ and $\vec{V}_d$.

It is shown in Reference 1 that the mean square deflection to be expected for a test particle of charge-to-mass ratio q/m is

$$< (\Delta\theta)^2 > = \frac{2\pi q^2}{m^2} \frac{L_o}{V_o^3} \int d\vec{k} \; < S_\perp >_{\vec{k}, -\vec{k}\cdot\vec{V}_o} , \qquad (1)$$

where $L_o \equiv d/\sin\gamma$ is the path length of the unperturbed trajectory which lies <u>inside</u> the plasma. $< S_\perp >_{\vec{k}\omega}$ is given by

$$< S_\perp >_{\vec{k}\omega} = \frac{2n_o e^2}{\pi} \frac{k_\perp^2}{k^5} \frac{\Sigma_j F_j (-\omega/k)}{|D^+ (\vec{k}, i\omega)|^2} , \qquad (2)$$

$$(\vec{k}_\perp \equiv \vec{k} - \vec{k} \cdot \vec{V}_o/V_o)$$

for a two-component plasma of singly charged ions and electrons of number density n_o. The distribution for the j^{th} species of particle (j = i for ions, j = e for electrons) is $f_j(\vec{v})$, and

$F_j(u) \equiv \int f_j(\vec{v})\, \delta(u - \vec{k}\cdot\vec{v}/k)\, d\vec{v}$ is the one-dimensional distribution in the component of velocity along the vector $\vec{k}$. The plasma dielectric function[2] is

$$D^+(\vec{k}, i\omega) = 1 - \lim_{\epsilon \to 0} \Sigma_j \frac{\omega_{p_j}^2}{k} \int_{-\infty}^{\infty} \frac{F_j'(u)\, du}{\omega + ku - i\epsilon} . \qquad (3)$$

The plasma frequency of the j^{th} component is $\omega_{p_j}^2 = 4\pi n_o e^2/m_j$.

Formulas (1), (2), and (3) in principle solve the problem, but the complicated nature of the spectral density $< S_\perp >_{\vec{k}\omega}$ makes the many definite integrals hard to do. The rest of this paper is devoted to evaluation of these formulas for the distributions

$$f_i(\vec{v}) = (2\pi V_i^2)^{-3/2} \exp(-\vec{v}^2/2V_i^2), \qquad (4a)$$

$$f_e(\vec{v}) = (2\pi V_e^2)^{-3/2} \exp(-(\vec{v} - \vec{V}_d)^2/2V_e^2), \qquad (4b)$$

where $V_i^2 = KT_i/m_i$, $V_e^2 = KT_e/m_e$, and the drift velocity V_d is less than the velocity required for the plasma to become electrostatically unstable. The case of most laboratory interest is that of $T_e \gg T_i$. Equations (4a) and (4b) give at once that

$$F_i(u) = (2\pi V_i^2)^{-1/2} \exp(-u^2/2V_i^2) \text{ and}$$

$$F_e(u) = (2\pi V_e^2)^{-1/2} \exp(-(u - \hat{k}\cdot\vec{V}_d)^2/2V_e^2),$$

$$\hat{k} \equiv \vec{k}/k .$$

The $\vec{k}$ integration is most expeditiously performed in spherical polar coordinates as shown in Fig. 2. Referred to Cartesian coordinates with x-axis along $\vec{V}_o$ and $\vec{V}_d$ in the xz plane,

$$\begin{aligned}
\vec{k} &= k\,(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi),\\
\vec{V}_d &= V_d\,(\cos\gamma,\ 0,\ \sin\gamma),\\
\vec{V}_o &= V_o\,(1,\ 0,\ 0),\\
\vec{k}\cdot\vec{V}_o &= k\,V_o\cos\theta,\\
\hat{k}\cdot\vec{V}_d &= V_d\,[\cos\theta\cos\gamma + \sin\theta\sin\gamma\sin\varphi],\\
d\vec{k} &= k^2\,dk\,d\,(\cos\theta)\,d\varphi\,.
\end{aligned}$$

As usual, the radial k-integration will diverge logarithmically because in calculating spectral densities for a plasma, the short range part of the Coulomb potential cannot readily be given proper treatment. Also as usual, we circumvent this by cutting off the integral at $k = k_o$, choosing k_o to be of the order of the inverse distance of closest approach of the test particle to a thermal particles of species j. This amounts to ignoring the occasional large-angle scattering events that result from zero impact parameter, and that become arbitrarily infrequent as the plasma parameter is lowered, relative to the part of the scattering that comes from $k < k_o$.

The integral in Eq. (1) therefore reduces to

$$\int d\vec{k} < S_\perp >_{\vec{k},\ -\vec{k}\cdot\vec{V}_o} = \frac{2n_o\ e^2}{\pi}\ \mathcal{J} \tag{5}$$

where

$$\mathcal{J} = \int_o^{k_o} k^3\ dk \int_{-1}^{1} d\ (\cos\ \theta)\ (1 - \cos^2\ \theta)$$

$$\int_o^{2\pi} d\varphi\ \frac{\Sigma_j\ F_j\ (V_o\ \cos\ \theta)}{(k^2 + \psi_r)^2 + \psi_i^2} \tag{6}$$

and $\psi\ (\theta,\ \varphi) = \psi_r + i\ \psi_i$, with real and imaginary parts

$$\psi_r = -\Sigma_j\ \omega_{p_j}^2\ P \int_{-\infty}^{\infty} \frac{F_j'\ (u)\ du}{u - V_o\ \cos\ \theta}\ , \tag{7a}$$

$$\psi_i = -\pi\ \Sigma_j\ \omega_{p_j}^2\ F_j'\ (V_o\ \cos\ \theta)\ . \tag{7b}$$

[In Eq. (7a), P means Cauchy principal value.]

Under most circumstances, the expression (6) is relatively straightforward to evaluate numerically, as shown in the next section. One exception to this occurs near the instability threshold, which is characterized by the existence of a sharp minimum in $|\ D^+\ (\vec{k},\ -i\vec{k}\cdot\vec{V}_o)\ | = k^{-2}\ [(k^2 + \psi_r)^2 + \psi_i^2\ ]^{1/2}$. The minimum eventually becomes steep enough to make numerical integration difficult. When it becomes sufficiently steep, a delta-function approximation enables one to write down the leading term in $\mathcal{J}$ in terms of the behavior of the integrand near the minimum.

The minimum value of the denominator[2,3] may occur at $k = 0$, or at finite values of k (when $\psi_r < 0$), depending upon T_e/T_i. The effect on $\mathcal{J}$ is more marked in the latter case (large T_e/T_i) because of the k^3 factor in the integrand, which cuts down the amount of available $\vec{k}$-space for a resonance near the origin.

The radial k-integration is essentially the same as one performed by Tidman and Eviatar,[4] and can be done analytically to give

$$\mathcal{J} = \int_{-1}^{+1} d\,(\cos\theta)(1 - \cos^2\theta) \int_0^{2\pi} d\varphi \left\{ 1/4 \;\ln\left[\frac{(\psi_r + k_o^2)^2 + \psi_i^2}{k_e^4} \right] - 1/4 \;\ln\left[\frac{\psi_r^2 + \psi_i^2}{k_e^4} \right] - \frac{\psi_r}{2\,|\psi_i|}\left[\tan^{-1} \frac{\psi_r + k_o^2}{|\psi_i|} - \tan^{-1} \frac{\psi_r}{|\psi_i|} \right] \right\} \Sigma_j\, F_j\, (V_o \cos\theta), \tag{8}$$

where $-\pi/2 < \tan^{-1} \leq \pi/2$ defines the branch of $\tan^{-1}$. The electron Debye wave number is $k_e^2 = \omega_{p_e}^2/V_e^2$. The next section is devoted to the evaluation of Eq. (8) for various values of T_e/T_i, V_o/V_i, γ, and V_d/V_c. We define V_c as the critical electron drift velocity required for the onset of the ion sound wave instability.

III. NUMERICAL RESULTS

We restrict ourselves to the case in which $V_e \gg V_o$, V_d, and V_i. Also k_o^2 will always be $\gg |\psi_r|$ and $|\psi_i|$. Under these restrictions, the first logarithm in (8) is well approximated by $1/4 \ \ell n \ (k_o^4/k_e^4) \equiv \ell n \ \Lambda$, say. Also, $F_e \ (V_o \cos \theta)$ is well approximated by $(2\pi \ V_e^2)^{-1/2}$, and $F_e' \ (V_o \cos \theta)$ by $-(2\pi \ V_e^2)^{-1/2} \cdot (u - V_d \ (\cos \theta \cos \gamma + \sin \theta \sin \gamma \sin \varphi)) \ V_e^{-2}$. The electron contribution to ψ_r is satisfactorily represented by k_e^2. The ion terms are <u>not</u> well approximated by their asymptotic forms, however, and the full Fried-Conte functions[2] must be used over much of the range of integration, since $V_o \cos \theta$ may be of the order of V_i.

With these simplifications, Eq. (8) may be reduced to the following two-dimensional definite integral

$$\mathcal{J} = 2\pi \ \ell n \ \Lambda \left[\frac{1}{V_o} + \frac{4/3}{\sqrt{2\pi} \ V_e} \right] - I, \tag{9}$$

where

$$I \equiv \frac{1}{\sqrt{2\pi} \ V_e} \int_0^{2\pi} d\varphi \int_{-1}^{1} dx \left\{ \frac{1}{4} \ \ell n \ (\psi_1^2 + \psi_2^2) + \frac{\psi_1}{2 \, |\psi_2|} \left(\frac{\pi}{2} - \tan^{-1} \frac{\psi_1}{|\psi_2|} \right) \right\} \left\{ (1 - x^2) \left(1 + \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{1/2} e^{-V_o^2 \ x^2/2V_i^2} \right) \right\} . \tag{10}$$

The symbols mean the following

$$\psi_1 = 1 + \frac{T_e}{T_i} Z \left(\frac{V_o \cos\theta}{\sqrt{2}\, V_i} \right),$$

$$Z(r) = \pi^{-1/2} P \int_{-\infty}^{\infty} \frac{y\, e^{-y^2}\, dy}{y - r},$$

$$\psi_2 = \sqrt{\frac{\pi}{2}} \left[\frac{T_e}{T_i} \frac{V_o x}{V_i} e^{-V_o^2 x^2/2V_i^2} \right.$$

$$\left. + \left(\frac{x}{V_e}\right)(V_o - V_d \cos\gamma) - \frac{V_d}{V_e}(1 - x^2)^{1/2} \sin\gamma \sin\varphi \right]$$

$$\equiv k_e^{-2} [f(x) - a(x) \sin\varphi].$$

The only φ-dependence of the integrand is the $\sin \varphi$ in the last term of ψ_2. The last expression defines the functions a (x) and f (x) for later use. In the cases considered, $V_o \ll V_e$, so $1/V_o$ dominates the first term in Eq. (9). Consistent with the fact that the ion integral is the dominant contribution and $V_o \gg V_i$, we choose $k_o = mV_o^2/2e^2$.

The isotropic case ($V_d = 0$) is considered first (Fig. 3). The most convenient quantity to plot is the dimensionless number Θ^2, defined by

$$\langle (\Delta\theta)^2 \rangle = \frac{8\pi q^2 e^2 n_o L_o}{m^2 V_o^4} \Theta^2$$

$$= \frac{8\pi q^2 e^2 n_o L_o}{m^2 V_o^4} \left\{ \ln\Lambda - \frac{V_o}{2\pi} I \right\}. \qquad (11)$$

That is, Θ^2 is proportional to the sum of what is ordinarily called the thermal scattering and the additional scattering due to any enhanced fluctuations which may be present. Note that a fixed value of Θ^2 means smaller total deflection for higher velocities V_o.

The result of the numerical integration for $V_d = 0$ and various ratios T_e/T_i is shown in Fig. 3 for different values of V_o. In all cases the ratio m_e/m_i is chosen to be 1/1836, so that we deal with hydrogen plasmas throughout. For reference purposes, the approximate formula of Tidman and Eviatar [Eq. (49), Ref. 4] is plotted on the same scale, with the factor $8\pi\, q^2\, e^2\, n_o\, L_o/m^2\, V_o^4$ taken out. Up to terms of $O\ (V_i/V_e)$, Tidman and Eviatar's formula is independent of V_o. In all cases, raising T_e/T_i at fixed T_i raises the level of the fluctuations and thus also the deflection. But the effect is not dramatic; raising T_e/T_i from 1 to 300 increases $< (\Delta\theta)^2 >$ by a factor $\lesssim 4$.

Figure 4 shows the variation of Θ^2 with angle of incidence γ for increasing test particle velocity V_o. The ratio of electron to ion temperature is held fixed at 100. The value of Θ^2 is peaked at a finite angle of the order of 60°, and the peak moves toward 90° as V_o increases. This peaking can be explained qualitatively as follows. The fluctuating electric field may be resolved into electrostatic waves of given wave number $\vec{k}$. These propagate, in the low-k part of the spectrum, with velocities of the order of the ionic sound wave speed, $\sqrt{KT_e/m_i}$, along $\vec{k}$. These are the fluctuations which are enhanced. In order for them to deflect a test particle efficiently, <u>two</u> conditions must be met: (1) The test particle must see a nearly time independent electric field when viewed from its own instantaneous rest-frame ($\hat{k} \cdot \vec{V}_o \cong \sqrt{KT_e/m_i}$), and (2) that k must correspond to the direction of propagation of the most enhanced fluctuations. In this geometry, the fluctuations are most enhanced for k parallel to $\vec{V}_d$. As V_o becomes $\gg \sqrt{KT_e/m_i}$ [it goes from 1 to 5 times $\sqrt{KT_e/m_i}$ in Fig. 4], the cosine of the angle between $\vec{V}_o$ and the direction of most enhanced k must go to zero to satisfy the first condition. In the limit of very large V_o, the peak will lie exactly at $\gamma = 90°$.

Figure 5 illustrates the effect of increasing the relative drift V_d at fixed T_e, T_i, and V_o. The critical drift velocity (numerically determined as the value of V_d for which ψ_2 first passes through zero with $\psi_1 < 0$) is found to lie between the values 5.48 and 5.49. [This is slightly smaller than Fried and Gould's value of $4\sqrt{2}$; part of the difference lies in our use of the asymptotic form for the electron Fried-Conte function.] By the time V_d/V_i gets to be 5.48, the integrand in I (Eq. (10)) has become so sharply peaked around the minimum value of $|\psi_2|$, that the two-dimensional integration becomes difficult.

The peak is to some extent offset by a large <u>positive</u> peak which occurs near $x = 0$. The volume under either peak is much greater than the difference, which contributes to Fig. 5.

For V_d within an infinitesimal distance of the instability threshold, the integral in Eq. (10) can be approximated by its contribution from the immediate neighborhood of $|\psi_2|_{min}$ only. The φ-integration can be done explicitly, and in the limit, the x-integrand goes over into a delta function, giving

$$\mathcal{J} \cong 2\pi^2 (1 - x_o^2)\ \psi_r\ (x_o)$$
$$\cdot \left\{ \Sigma_j\ F_j\ (V_o\ x_o) \right\} C^{-1/2}\ (x_o)\ \ell n \left\{ \frac{1}{\epsilon\ (x_o)} \right\} , \tag{12}$$

where we have explicitly separated the φ-dependence of ψ_i according to

$$\psi_i\ (x, \varphi) \equiv f\ (x) - a\ (x) \sin \varphi,$$
$$\epsilon\ (x_o) \equiv f^2\ (x_o) - a^2\ (x_o),$$
$$C\ (x_o) \equiv \frac{1}{2!} \frac{\partial^2 \epsilon\ (x_o)}{\partial x_o^2} .$$

x_o is the location of the minimum of $\epsilon\ (x_o)$, whose vanishing defines the instability threshold. It is simplest to determine the value of x_o for which ϵ takes on its minimum by a numerical search.

Unfortunately, the delta-function approximation (12) is likely to be useful only as a "display" formula. The reason is the slow (logarithmic) divergence in $\ln \left\{ 1/\epsilon\ (x_o) \right\}$ at the instability boundary. Near the instability boundary, $\epsilon\ (x_o)$ varies proportionately to $|V_d - V_c|$. To increase $\ln\ [V_c/\ |V_d - V_c|\ ]$ by an order of magnitude, $|V_d - V_c|/V_c$ must be decreased by a factor of e^{-10}. Since in Fig. 5, one is already within 0.1 per cent of the critical drift velocity, this means one has to be able to control V_d to one part in a million, or better, to increase the scattering by an order of magnitude, an experimental absurdity. The slow character of the divergence makes drastically enhanced scattering extremely unlikely. This is in sharp contrast to the electromagnetic wave case,[7] where the divergence is $O\ (1/\epsilon\ (x_o))$.

IV. DISCUSSION AND SUMMARY

Deflections of a test particle by a plasma may be enhanced by raising the electron temperature or the electron drift, as previously noted by Tidman and Eviatar. The purpose of this calculation has been primarily to make this conclusion more quantitative for laboratory plasmas. The enhancement of particle scattering is less drastic than in the case of electromagnetic wave scattering, but is probably measurable, even so. The peaked deflection near $\gamma = 90°$ shown in Fig. 4 is a particularly intriguing experimental possibility. The theoretical divergence in the deflections at the threshold of linear instability is probably of no experimental interest; the extreme precision with which the critical drift must be approached is probably experimentally prohibitive.

We have avoided possible errors in the usual ninety-degree deflection times by casting the theory entirely in terms of small-angle deflections, as discussed in the Appendix.

FOOTNOTES

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APPENDIX

REMARKS ON DEFLECTION TIMES; COMPARISON WITH THE THEORY OF BROWNIAN MOTION

Our quantity $< (\Delta\theta)^2 >$ is closely related to the "ninety-degree deflection time", τ_D, used by Tidman and Eviatar.[4,8] τ_D is defined for any test particle distribution $f_t\ (\vec{v},\ t)$ which obeys the linear Fokker-Planck equation

$$\frac{\partial f_t}{\partial t} = - \frac{\partial}{\partial \vec{v}} \cdot (\vec{F}\ f_t) + \frac{1}{2}\ \frac{\partial^2}{\partial \vec{v}\ \partial \vec{v}} : (\vec{\vec{T}}\ f_t) \tag{A1}$$

and reduces to $f_t = \delta\ (\vec{v} - \vec{V}_o)$ at $t = 0$. It is defined[8] by

$$\tau_D \equiv \left. \frac{< v^2 >}{\partial < v_\perp^2 >/\partial t} \right]_{t=0} ,$$

where $< v^2 > \equiv \int f_t\ \vec{v}^2\ d\vec{v}$, and where $< v_\perp^2 > \equiv \int f_t\ (\vec{v} - \vec{V}_o \cdot \vec{v}/V_o)^2\ d\vec{v}$. In terms of the Fokker-Planck coefficients $\vec{F}$ and $\vec{\vec{T}}$ (assumed known), this is readily shown to be

$$\tau_D = \frac{V_o^2}{T_\perp\ (\vec{V}_o)} , \tag{A2}$$

where $T_\perp\ (\vec{V}_o)$ is the sum of the two diagonal components of $\vec{\vec{T}}\ (\vec{V}_o)$

which are perpendicular to $\vec{V}_o$. The expression for $\overset{\Rightarrow}{T}$ appropriate to this case has been calculated by Tidman, Guernsey, and Montgomery[9] and when it is inserted into the expression just given for τ_D, one gets

$$\tau_D = t_o / < (\Delta\theta)^2 > , \tag{A3}$$

with $< (\Delta\theta)^2 >$ given by our Eq. (11), as the relation between τ_D and $< (\Delta\theta)^2 >$. [t_o is L_o/V_o.]

However, the <u>interpretation</u> we are attaching to $< (\Delta\theta)^2 >$ is somewhat different. It is by hypothesis a <u>small</u> angle, $< (\Delta\theta)^2 > \ll (\pi/2)^2$. It is an open question how accurately formula (A2) actually does predict the time required to deflect a test particle through <u>large</u> angles, $\cong \pi/2$, in the plasma case. A conclusive answer to this question would require much more knowledge of the solution of the Fokker-Planck equation for a test particle in a plasma than we now possess. But an interesting test of this method [of extrapolating the $t = 0$ values of $< v^2 >/\partial < v_\perp^2 >/\partial t$ to find ninety-degree deflection times] is provided by a <u>soluble</u> example of a Fokker-Planck equation, the equation of Brownian motion.[10] In this case $\vec{F} = -\beta\vec{v}$, $\overset{\Rightarrow}{T} = 2q \overset{\Rightarrow}{1}$, q and β are constants, and $q/\beta = KT/m$, where T is the temperature of the background medium. The solution of (A1) which reduces to $\delta (\vec{v} - \vec{V}_o)$ at $t = 0$ is, for this case,

$$f_t\,(\vec{v},\,t) \;=\; \left[\frac{\beta}{2\pi q\,(1 - e^{-2\beta t})}\right]^{3/2}$$

$$\exp\left[\frac{-\beta\,(\vec{v} - \vec{V}_o\, e^{-2\beta t})^2}{2q\,(1 - e^{-2\beta t})}\right] \;. \tag{A4}$$

This f_t leads immediately to the relations

$$< v^2 > \;=\; V_o^2\, e^{-2\beta t} + (3q/\beta)\,(1 - e^{-2\beta t}) \;\xrightarrow[t \to o]{}\; V_o^2 \,, \tag{A5}$$

$$< v_\perp^2 > \;=\; \frac{2q}{\beta}\;(1 - e^{-2\beta t}) \;\xrightarrow[t \to o]{}\; 0 \;, \tag{A6}$$

and using our definition of τ_D, for this case,

$$\tau_D \;=\; V_o^2/4q \;. \tag{A7}$$

The angular deflection may be defined by

$$\left\{\tan < (\Delta\theta)^2 >^{1/2}\right\}^2 \;\equiv\; \frac{< v_\perp^2 >}{< v_\parallel^2 >}$$

$$= \left\{\frac{1}{2} + \frac{\beta V_o^2}{2q}\;\frac{1}{e^{2\beta t} - 1}\right\}^{-1} \;. \tag{A8}$$

For small deflections (small t), Eq. (A8) gives $< (\Delta\theta)^2 > \;\simeq\; 4\, qt/V_o^2$, in precise agreement with (A3). However, if we ask for the value $t_{90°}$ necessary for a <u>large</u> deflection,

$< v_\perp^2 > \ \tilde{=} \ < v_\parallel^2 >$, we get $t_{90^\circ} \ \tilde{=} \ \frac{1}{2\beta} \ln \left(1 + \frac{\beta V_o^2}{q}\right)$, which may disagree by orders of magnitude, for V_o large, with Eq. (A3).

We infer from this example that: (i) $t = o$ values of Fokker-Planck coefficients are more reliable guides to small angle deflection times than to ninety degree deflection times; and (ii) more fundamental work is needed (perhaps numerical) on the test particle Fokker-Planck equation in a plasma before one can make convincing statements about large angle deflection times.

FIGURE CAPTIONS

Figure 1. Proposed Experiment (Idealized). The plasma is uniform in the rectangular solid and the electrons drift with velocity $\vec{V}_d$ along its length. The test particles of velocity $\vec{V}_o$ strike the plasma on the front face making an angle γ with $\vec{V}_d$. The test particles are deflected through a small angle $< (\Delta\theta)^2 >^{1/2}$ and are detected after they emerge from the back face.

Figure 2. Geometry of the Wave Number Integration. γ is fixed, but θ and φ must be integrated over all possible values.

Figure 3. Normalized Deflection, $\bar{\delta}^2$, vs T_e/T_i. Two values of the test particle velocity are shown (V_o/V_i = 10 and 20). The formula (49) of Tidman and Eviatar (which is independent of V_o as long as $V_o/V_e \ll 1$) is shown as a dashed line.

Figure 4. Variation of Normalized Deflection as a Function of Test Particle Velocity. Note that because of the factor V_o^{-4} in Eq. (11), the total deflection $< (\Delta\theta)^2 >$ goes down as V_o increases. V_d/V_i is about 5.48 for all three plots.

Figure 5. Variation of Normalized Deflection as Drift Velocity Approaches Unstable Value. In this approximation, the unstable value of V_d/V_i is less than 5.49.

G 67-566

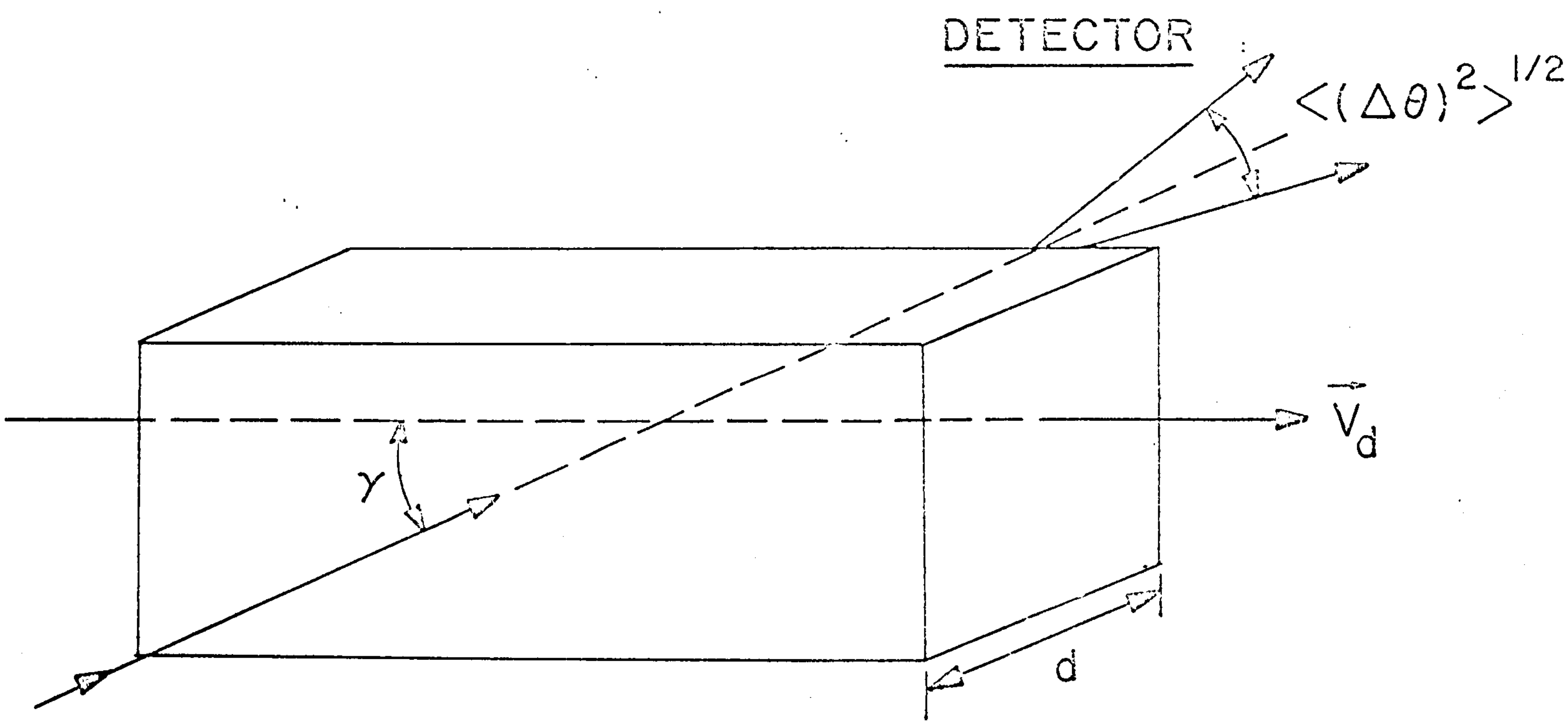


FIGURE 1

G 67-567

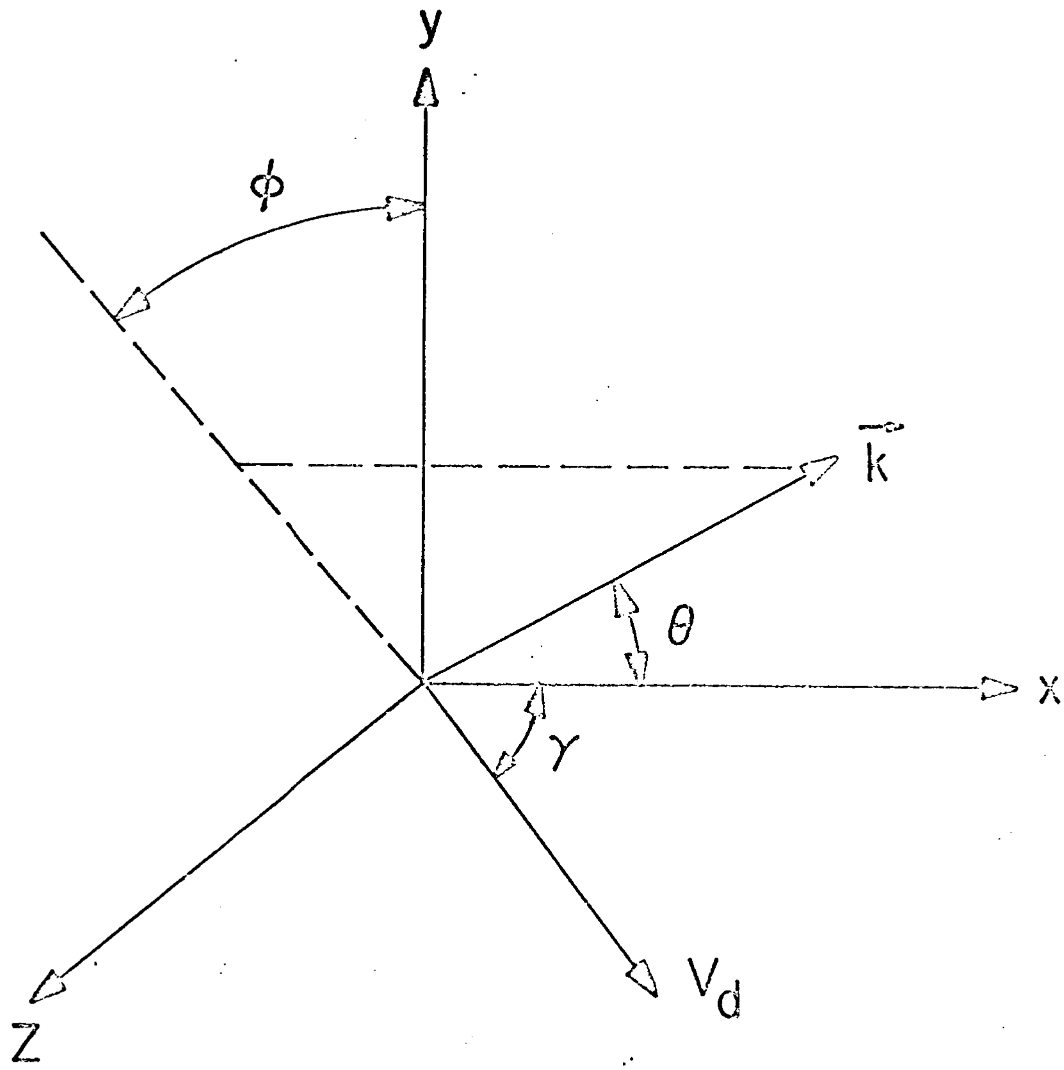


FIGURE 2

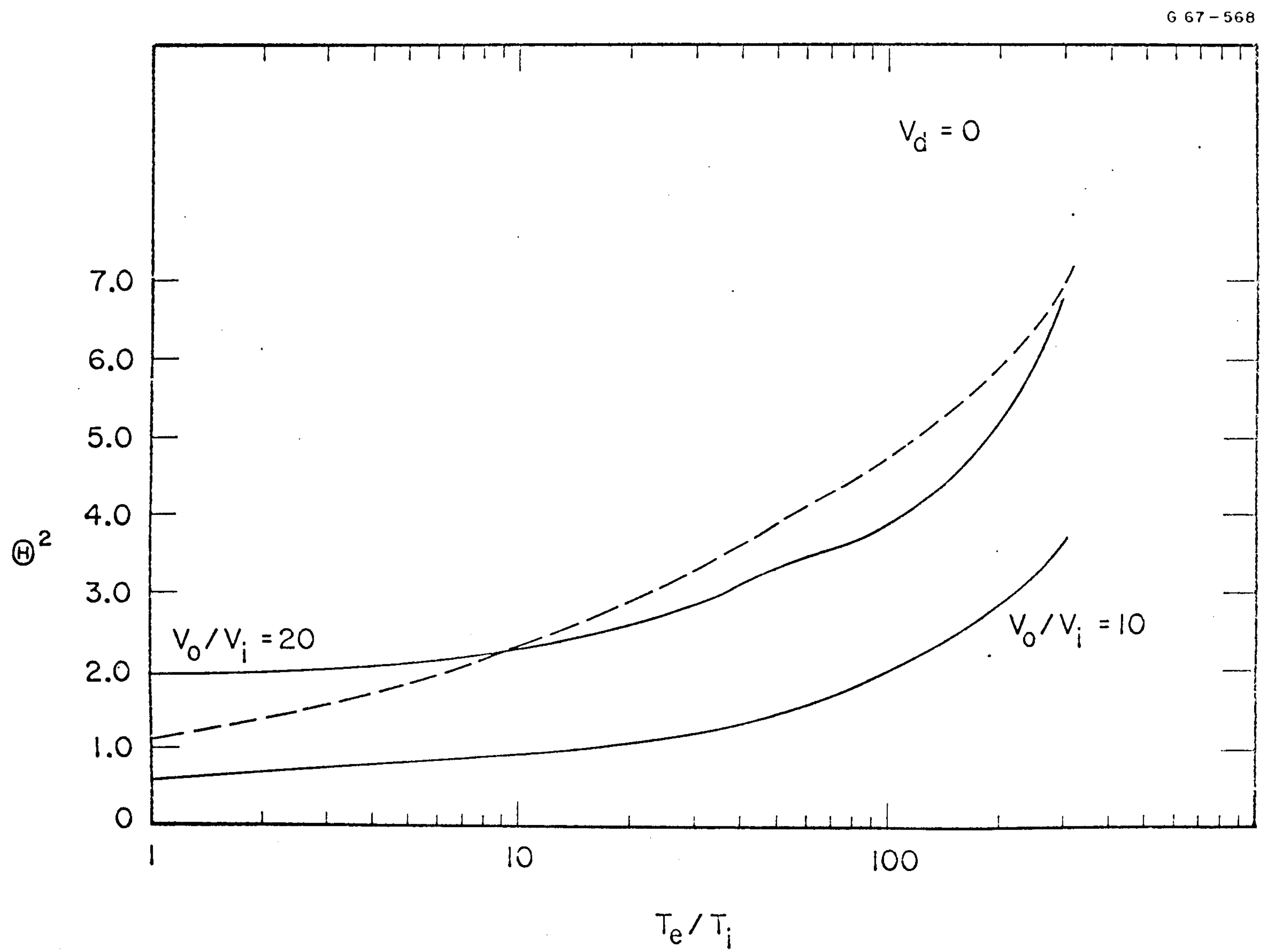
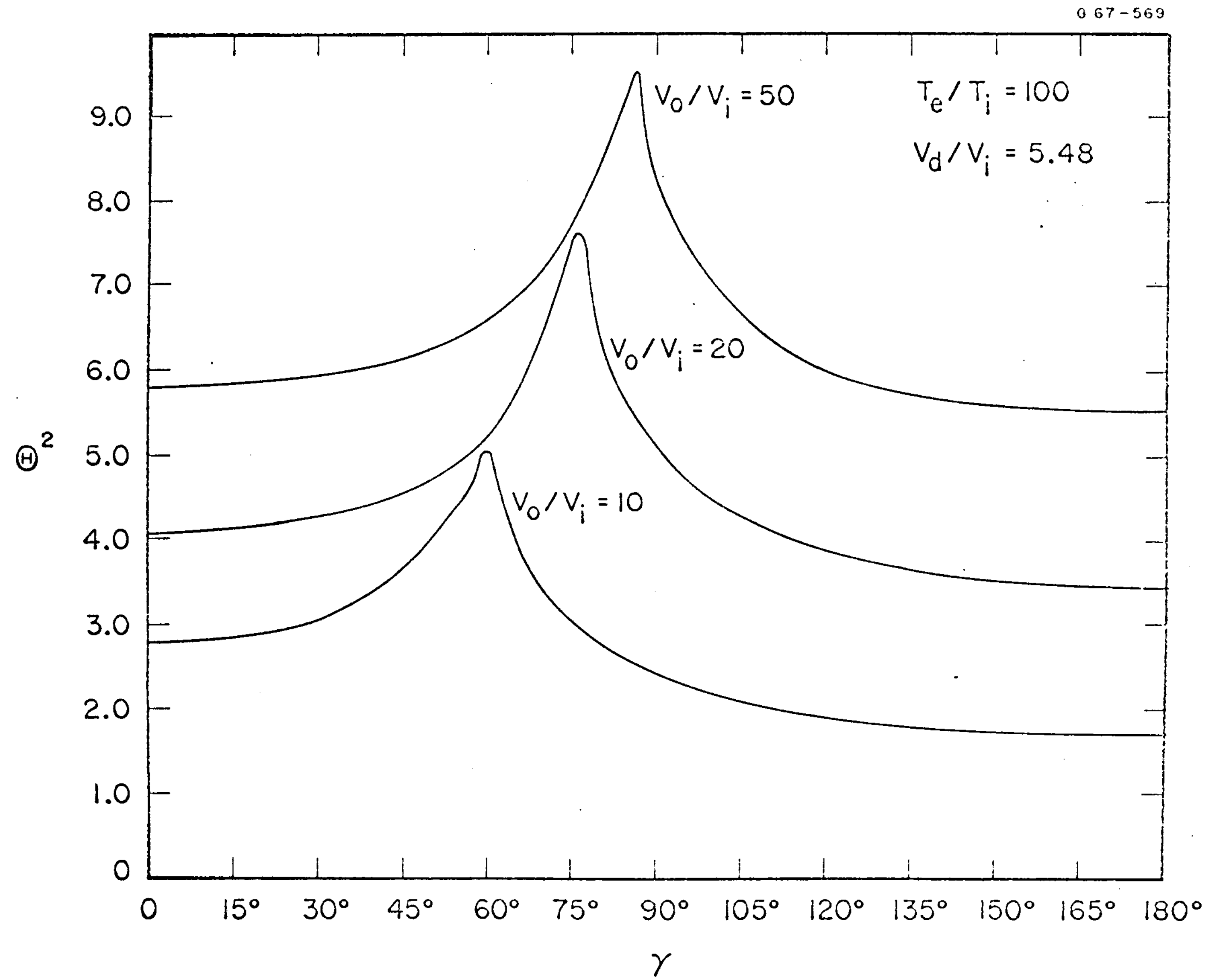

G 67-568
$V_d = 0$
$V_0/V_i = 20$
$V_0/V_i = 10$
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
Θ^2
1
10
100
T_e/T_i

FIGURE 3



0 67-569
Vo / Vi = 50
Vo / Vi = 20
Vo / Vi = 10
Te / Ti = 100
Vd / Vi = 5.48
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
0
15°
30°
45°
60°
75°
90°
105°
120°
135°
150°
165°
180°

FIGURE 4

G 67 - 570

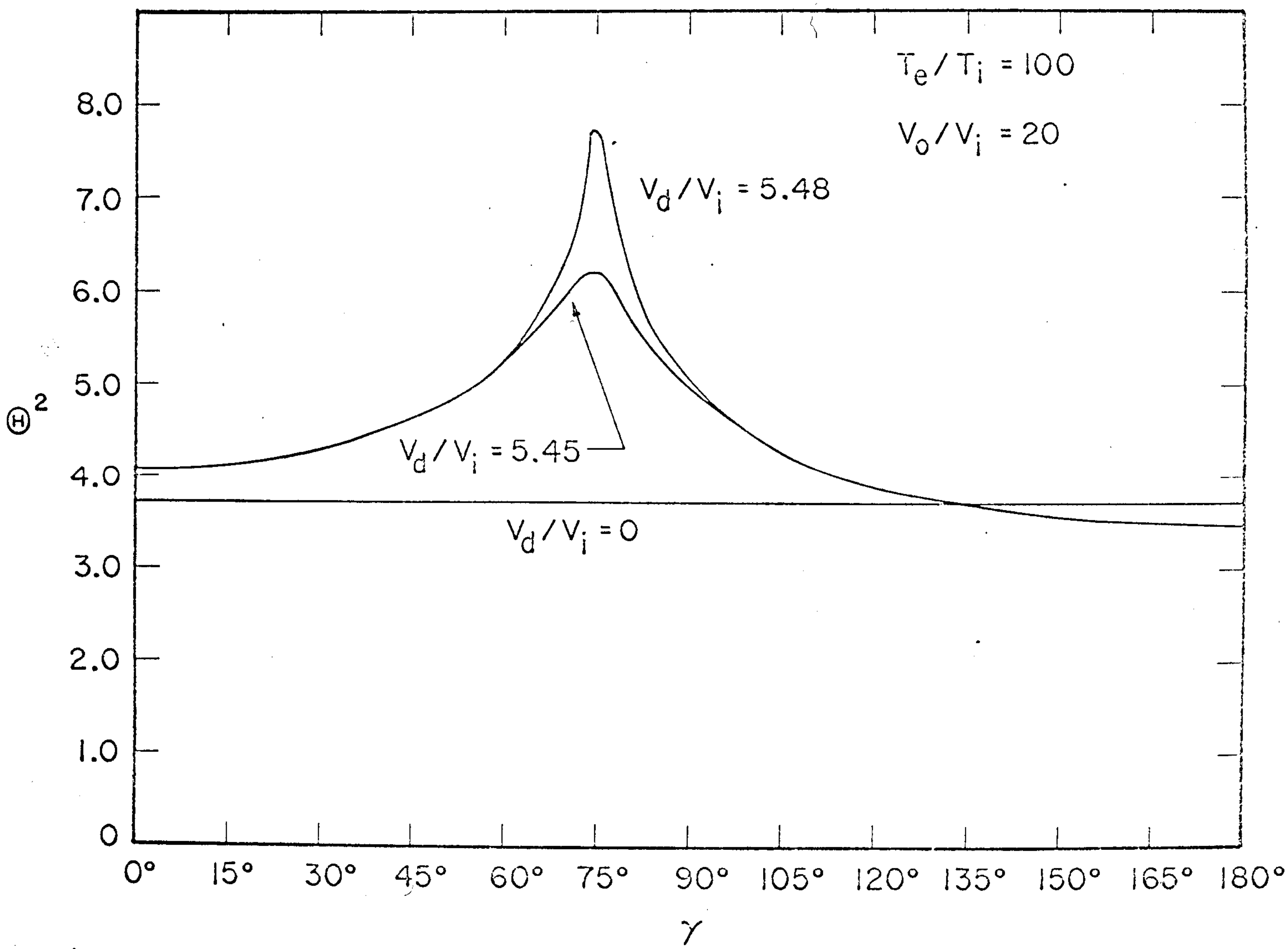


FIGURE 5